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Contact Hamiltonian mechanics

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ABSTRACT

In this work we introduce contact Hamiltonian mechanics, an extension of symplectic Hamiltonian mechanics, and show that it is a natural candidate for a geometric description of non-dissipative and dissipative systems. For this purpose we review in detail the major features of standard symplectic Hamiltonian dynamics and show that all of them can be generalized to the contact case.

1. Introduction

The Hamiltonian formulation of classical mechanics is a very useful tool for the description of mechanical systems due to its remarkable geometrical properties, and because it provides a natural way to extend the classical theory to the quantum context by means of standard quantization. However, this formulation exclusively describes isolated systems with reversible dynamics, while real systems are constantly in interaction with an environment that introduces the phenomena of dissipation and irreversibility. Therefore a major question is whether it is possible to construct a classical mechanical theory that not only contains all the advantages of the Hamiltonian formalism, but also takes into account the effects of the environment on the system.

Several programmes have been proposed for this purpose (see e.g. [1] for a recent review). For example, one can introduce *stochastic dynamics* to model the effect of fluctuations due to the environment on the system of interest. This leads to stochastic equations of the Langevin or Fokker–Planck

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type with diffusion terms [2,3]. A different although related approach is the *system-plus-reservoir* technique, in which the system of interest is coupled to an environment (usually modeled as a collection of harmonic oscillators). The system and the environment together are considered as an isolated Hamiltonian system and after averaging out the environmental degrees of freedom one obtains the equations of motion for the system of interest, including dissipative terms. This is the case for example of the Caldeira–Laggett formalism [4–6]. An alternative approach is to propose *effective Hamiltonians* with an explicit time dependence that reproduce the correct Newtonian equation, including the dissipative forces. A famous example is the Caldirola–Kanai (CK) model [7–9]. Another proposal based on a nonconservative action principle, allows for time-irreversible processes, such as dissipation, to be included at the level of the action [10,11]. Finally, a more geometrical attempt towards the description of dissipative systems is given by the so-called *bracket formulation* of dynamical systems [12]. Here one generalizes the standard Poisson bracket to a noncanonical Poisson bracket and exploits the algebraic properties of the latter to include dissipation. The literature on all these proposals is very extensive and it is not our purpose here to review them in detail. We refer the interested reader to the standard references cited above and references therein.

Here we discuss a new proposal which consists in extending the symplectic phase space of classical mechanics by adding an extra dimension, thus dealing with a contact manifold instead of a symplectic one. Contact geometry arises naturally in mechanics. First of all, in describing mechanical systems where the Hamiltonian function explicitly depends on time, one usually appeals to an extended phase space, the additional dimension being time, endowed with the Poincaré-Cartan 1-form, which defines a contact structure on the extended space [13–15]. Besides, the time-dependent Hamilton–Jacobi theory is naturally formulated in this extended phase space [16,17]. Furthermore, it has recently been argued that symmetries of the contact phase space can be relevant for a (non-canonical) quantization of nonlinear systems [18].

In this work we consider the phase space of any (time-independent) mechanical system (either non-dissipative or dissipative) to be a contact manifold, but we take a different route from previous works. In fact, there are two main differences between our proposal and the previous ones. First, we do not assume that the additional dimension is time, letting the additional dimension be represented by a non-trivial dynamical variable. Second, we derive the equations of motion for the system from *contact Hamiltonian dynamics*, which is the most natural extension of symplectic Hamiltonian dynamics [14].

Contact Hamiltonian dynamics has been used already in thermodynamics (both equilibrium and not [19–24]) and in the description of dissipative systems at the mesoscopic level [25]. Furthermore, it has been recently introduced in the study of mechanical systems exchanging energy with a reservoir [26,27]. However, a detailed analysis of the dynamics of mechanical systems and a thorough investigation of the analogy with standard symplectic mechanics have never been pursued before. We show that the advantages of contact Hamiltonian mechanics are that it includes within the same formalism both non-dissipative and dissipative systems, giving a precise prescription to distinguish between them, that it extends canonical transformations to contact transformations, thus offering more techniques to find the invariants of motion and to solve the dynamics, and that it leads to a contact version of the Hamilton–Jacobi equation. We argue that these additional properties play a similar role as their symplectic counterparts for dissipative systems.

The structure of the paper is as follows: in Section 2, in order to make the paper self-contained, we review the main aspects of the standard mechanics of non-dissipative systems, with emphasis on the symplectic geometry of the phase space and the Hamilton–Jacobi formulation. In Section 3 the same analysis is extended to the case of contact Hamiltonian systems and it is shown by some general examples that this formulation reproduces the correct equations of motion for mechanical systems with dissipative terms. Besides, an illustrative example (the damped parametric oscillator) is worked out in detail in this section in order to show the usefulness of our method. Section 4 is devoted to a summary of the results and to highlight future directions. In particular, we discuss a possible extension of our formalism to quantum systems. Finally, in Appendices A and B we provide respectively a derivation of the invariants of the damped parametric oscillator and a constructive proof of the equivalence between the contact Hamilton–Jacobi equation and the contact Hamiltonian dynamics.

Before starting, let us fix a few important notations that are used throughout the text. Both symplectic mechanics of conservative systems and contact mechanics of dissipative systems are presented first in a coordinate-free manner and then in special local coordinates – canonical and contact coordinates – labelled as (q^a, p_a) and (q^a, p_a, S) respectively. Moreover, the symplectic phase space is always indicated by Γ, while the contact phase space by $\mathcal{T}$. The extension of any geometric object to a quantity that explicitly includes time as an independent variable is always indicated with a superscript E over the corresponding object (e.g. Γ^{E}). Finally, we always use the notation H for the usual symplectic Hamiltonian function and $\mathscr{H}$ for the corresponding contact analogue.

2. Symplectic mechanics of non-dissipative systems

The description of isolated mechanical systems can be given in terms of the Hamiltonian function and of Hamilton's equations of motion in the phase space, which has a natural symplectic structure. In this section we review Hamiltonian dynamics in the symplectic phase space, in order to compare it with the generalization to the contact phase space that is given in the next section.

2.1. Time-independent Hamiltonian mechanics

The phase space of a conservative system is the cotangent bundle of the configuration manifold, which is a $2n$-dimensional manifold Γ. Such manifold is naturally endowed with a canonical 1-form α, whose exterior derivative $\Omega = \mathrm{d}\alpha$ is non-degenerate, and therefore defines the standard symplectic form on Γ. Given a Hamiltonian function H on Γ, Hamilton's equations of motion follow from

$$-\mathrm{d}H = \Omega(X_H), \tag{1}$$

with X_H the *Hamiltonian vector field* defining the evolution of the system. By a theorem of Darboux, one can always find local coordinates (q^a, p_a) with $a = 1, \ldots, n$ – called *canonical coordinates* – in which the canonical form is expressed as

$$\alpha = p_a \mathrm{d}q^a, \tag{2}$$

where here and in the following Einstein's summation convention over repeated indices is assumed. In such coordinates

$$\Omega = \mathrm{d}\alpha = \mathrm{d}p_a \wedge \mathrm{d}q^a \tag{3}$$

and from (1) it follows that the Hamiltonian vector field reads

$$X_H = -\frac{\partial H}{\partial q^a}\frac{\partial}{\partial p_a} + \frac{\partial H}{\partial p_a}\frac{\partial}{\partial q^a}. \tag{4}$$

Usually, the canonical coordinates q^a and p_a correspond to the particles' generalized positions and momenta. From (4) the equations of motion take the standard Hamiltonian form

$$\dot{q}^a = \frac{\partial H}{\partial p_a}, \qquad \dot{p}_a = -\frac{\partial H}{\partial q^a}. \tag{5}$$

A system whose evolution is governed by (5) is usually called a *Hamiltonian system*. The time evolution of any (not explicitly time-dependent) function $G \in C^\infty(\Gamma)$ is determined by the phase space trajectories generated by the Hamiltonian vector field X_H, that is

$$\frac{\mathrm{d}G}{\mathrm{d}t} = X_H[G] = \Omega(X_H, X_G) = \{G, H\}_{(q^a, p_a)}, \tag{6}$$

where we have introduced the notation $\{G, H\}_{(q^a, p_a)}$ for the standard *Poisson bracket* between the two functions G and H, which in canonical coordinates reads

$$\{G, H\}_{(q^a, p_a)} = \frac{\partial G}{\partial q^a}\frac{\partial H}{\partial p_a} - \frac{\partial G}{\partial p_a}\frac{\partial H}{\partial q^a}. \tag{7}$$

Eqs. (6) and (7) imply immediately that H is a first integral of the flow, that is, energy is conserved. In addition, any function commuting with H is also a first integral.

2.2. Canonical transformations and Liouville's theorem

Canonical transformations are an extremely important tool in classical mechanics, as they are strongly related to the symmetries and to the conserved quantities of the system and hence they are useful to simplify the equations of motion. They can be classified in *time-independent transformations*, the ones that preserve the form of the Hamiltonian function, and *time-dependent transformations*, which include time in the transformation and therefore are properly defined in an extended phase space. Here we consider time-independent transformations only. Time-dependent transformations are introduced below.

Canonical transformations are change of coordinates in the phase space that leave Hamilton's equations (5) invariant. From (1), this amounts at finding a change of coordinates in the phase space that preserves the symplectic form Ω [14]. This definition immediately yields a way to check whether a coordinate transformation is canonical. Given the transformation $(q^a, p_a) \to (Q^a, P_a)$, invariance of Ω implies the following conditions

$$\left\{Q^a, Q^b\right\}_{(q^i,p_i)} = 0, \quad \{P_a, P_b\}_{(q^i,p_i)} = 0, \quad \left\{Q^a, P_b\right\}_{(q^i,p_i)} = \delta^a_b. \tag{8}$$

As a consequence, canonical transformations also leave the canonical form α invariant up to an exact differential, that is

$$p_a\, \mathrm{d}q^a = P_a\, \mathrm{d}Q^a + \mathrm{d}F_1, \tag{9}$$

where $F_1(q^a, Q^a)$ is called the *generating function* of the canonical transformation and obeys the relations

$$p_a = \frac{\partial F_1}{\partial q^a}, \quad P_a = -\frac{\partial F_1}{\partial Q^a}. \tag{10}$$

Furthermore, as Ω^n is the volume element of the phase space, then it follows that canonical transformations preserve the phase space volume. A particular case of canonical transformations is the Hamiltonian evolution (5). In fact, symplectic Hamiltonian vector fields X_H are the infinitesimal generators of canonical transformations. Therefore Liouville's theorem

$$\pounds_{X_H}\, \Omega^n = 0 \tag{11}$$

follows directly, where $\pounds_{X_H}$ is the Lie derivative along the Hamiltonian vector field X_H [14].

2.3. Time-dependent Hamiltonian systems

For mechanical systems whose Hamiltonian depends explicitly on time the Eq. (1) are no longer valid, since the differential of the Hamiltonian depends on time. Moreover, also in the case of time-independent systems, it is useful to consider time-dependent canonical transformations, for which the differential of the corresponding generating functions does not satisfy the canonical condition (9). In order to deal with time-dependent systems and time-dependent canonical transformations, one usually extends the phase space with an extra dimension representing time. The extended phase space $\Gamma^{\mathrm{E}} = \Gamma \times \mathbb{R}$ is therefore a $(2n+1)$-dimensional manifold endowed with a 1-form

$$\eta_{\mathrm{PC}} = p_a \mathrm{d}q^a - H \mathrm{d}t, \tag{12}$$

called the *Poincaré-Cartan* 1*-form*, where the Hamiltonian H can either depend explicitly on time or not.[1] Then one proceeds to define a dynamics on Γ^{E} that correctly extends Hamiltonian dynamics to the case where the Hamiltonian depends explicitly on time. A direct calculation shows that the condition

$$\mathrm{d}\eta_{\mathrm{PC}}(X_H^{\mathrm{E}}) = 0 \tag{13}$$

[1] Notice that $(\Gamma^{\mathrm{E}}, \eta_{\mathrm{PC}})$ is a contact manifold (cf. Section 3.1), but it is not the standard (natural) contactification of (Γ, Ω) (see [28]), since η_{PC} depends on H and hence on the system.

is satisfied if and only if the vector field X_H^{E} in these coordinates takes the form

$$X_H^{\mathrm{E}} = X_H + \frac{\partial}{\partial t}, \tag{14}$$

where X_H is given by (4). Therefore the equations of motion for this vector field read

$$\dot{q}^a = \frac{\partial H}{\partial p_a}, \qquad \dot{p}_a = -\frac{\partial H}{\partial q^a}, \qquad \dot{t} = 1, \tag{15}$$

which are just Hamilton's equations (5), augmented with the trivial equation $\dot{t} = 1$. This makes clear that Hamilton's equation in the extended phase space (15) are equivalent to the condition (13). It follows that the evolution of an arbitrary function $G \in C^\infty(\Gamma^{\mathrm{E}})$ is given by

$$\frac{\mathrm{d}G}{\mathrm{d}t} = \{G, H\}_{(q^a, p_a)} + \frac{\partial G}{\partial t}, \tag{16}$$

and consequently for time-dependent Hamiltonian systems the Hamiltonian itself is not conserved.

Now let us study time-dependent canonical transformations and their generating functions. To do so, we need to find a change of coordinates from (q^a, p_a, t) to (Q^a, P_a, t) that leaves the form of the extended Hamilton's equations (15) unchanged. Since in condition (13) only the differential of η_{PC} is involved, we find out that we can make a transformation that changes η_{PC} by the addition of an exact differential, so that Eq. (13) is not affected. Let us consider such transformation and write

$$p_a \mathrm{d}q^a - H\mathrm{d}t - (P_a \mathrm{d}Q^a - K\mathrm{d}t) = \mathrm{d}F_1, \tag{17}$$

where K is a function on Γ^{E} which is going to be the new Hamiltonian function after the transformation. Let us further assume that we can choose coordinates in which Q^a and q^a are independent, so that the independent variables in (17) are (q^a, Q^a, t). We rewrite (17) as

$$\left(p_a - \frac{\partial F_1}{\partial q^a}\right) \mathrm{d}q^a - \left(P_a + \frac{\partial F_1}{\partial Q^a}\right) \mathrm{d}Q^a + \left(K - H - \frac{\partial F_1}{\partial t}\right) \mathrm{d}t = 0, \tag{18}$$

which implies that the *generating function* of the canonical transformation $F_1(q^a, Q^a, t)$ satisfies the relations

$$p_a = \frac{\partial F_1}{\partial q^a}, \quad P_a = -\frac{\partial F_1}{\partial Q^a}, \quad K = H + \frac{\partial F_1}{\partial t}. \tag{19}$$

Hamilton's equations (15) in the new coordinates can be written as

$$\dot{Q}^a = \frac{\partial K}{\partial P_a}, \qquad \dot{P}_a = -\frac{\partial K}{\partial Q^a}, \qquad \dot{t} = 1, \tag{20}$$

with K the new Hamiltonian.

Systems with explicit time dependence are used for the effective description of dissipative systems within the Hamiltonian formalism. The idea is to introduce a convenient time dependence into the Hamiltonian so that it reproduces the phenomenological equations of motion with energy dissipation. As an example, let us consider the approach by Caldirola [7] and Kanai [8] for a 1-dimensional dissipative system with a friction force linear in the velocity. This model considers the time-dependent Hamiltonian

$$H_{\mathrm{CK}} = e^{-\gamma t} \frac{p_{\mathrm{CK}}^2}{2m} + e^{\gamma t} V(q_{\mathrm{CK}}), \tag{21}$$

where p_{CK} and q_{CK} are the canonical coordinates in phase space, which are related to the physical positions and momenta by the non-canonical transformation

$$p_{\mathrm{CK}} = e^{\gamma t} p, \quad q_{\mathrm{CK}} = q. \tag{22}$$

It is easy to show that Hamilton's equations (15) for H as in (21) give the correct equation of motion for the position including the friction force, i.e. the damped Newton equation

$$\ddot{q} + \gamma\,\dot{q} + \frac{1}{m}\frac{\partial V(q)}{\partial q} = 0. \tag{23}$$

However, although this model reproduces the correct phenomenological equation of motion, it has the drawback that in order to describe dissipative systems one needs to take into account the non-canonical relationship (22) between canonical and physical quantities. As a consequence, at the quantum level this model has generated quite a dispute on whether it can describe a dissipative system without violating the Heisenberg uncertainty principle; we refer to e.g. the discussion in [29–34] and references therein.

2.4. Hamilton–Jacobi formulation

The Hamilton–Jacobi formulation is a powerful tool which enables to re-express Hamilton's equations in terms of a single partial differential equation whose solution, a function of the configuration space, has all the necessary information to obtain the trajectories of the mechanical system. Moreover, this formulation gives rise to a new and more geometric point of view that allows to relate classical mechanics with wave phenomena and thus with quantum mechanics.

The Hamilton–Jacobi equation can be introduced as a special case of a time-dependent canonical transformation (19). Consider the case in which the new Hamiltonian K vanishes and write the generating function F_1 in this particular case as S. Using (19), we can write the *Hamilton–Jacobi equation*

$$H\left(q^a, \frac{\partial S}{\partial q^a}, t\right) = -\frac{\partial S}{\partial t}. \tag{24}$$

A complete solution $S(q^a, t)$, called *Hamilton's principal function*, determines completely the dynamics of the system [15]. Besides, since $K = 0$ for such transformation, it is clear from (20) that Hamilton's equations in the new coordinates read

$$\dot{Q}^a = 0, \qquad \dot{P}_a = 0. \tag{25}$$

Therefore, the new system of coordinates moves along the Hamiltonian flow. In fact, the functions $Q^a(q^i, p_i, t)$ and $P_a(q^i, p_i, t)$ are (generalized) Noether invariants associated with the Noether symmetries $\partial/\partial P_a$ and $\partial/\partial Q^a$ respectively [18]. Finally, the time derivative of Hamilton's principal function is given by

$$\dot{S} = \frac{\partial S}{\partial q^a}\dot{q}^a + \frac{\partial S}{\partial t} = p_a\dot{q}^a - H \tag{26}$$

where in the second identity we have used both (19) and (24). Since the right hand side of (26) is the Lagrangian of the system, one concludes that

$$S(q^a, t) = \int L(q^a, \dot{q}^a, t)\mathrm{d}t, \tag{27}$$

i.e. that Hamilton's principal function is the action, up to an undetermined additive constant [15].

3. Contact mechanics of dissipative systems

So far we have only reviewed the standard Hamiltonian description of mechanical systems. In this section we introduce the formalism of contact Hamiltonian mechanics and show that it can be applied to describe both non-dissipative and dissipative systems. Some of the material in Sections 3.1 and 3.3 has been already presented in [26,27].

3.1. Time-independent contact Hamiltonian mechanics

A *contact manifold* $\mathcal{T}$ is a $(2n+1)$-dimensional manifold endowed with a 1-form η, called the *contact form*, that satisfies the condition [28]

$$\eta \wedge (\mathrm{d}\eta)^n \neq 0. \tag{28}$$

The left hand side in (28) provides the *standard volume form* on $\mathcal{T}$, analogously to Ω^n for the symplectic case. Hereafter we assume that the phase space of time-independent mechanical systems (both dissipative and non-dissipative) is a contact manifold, called the *contact phase space*,[2] and that the equations of motion are always given by the so-called contact Hamiltonian equations. We show that in this way one can construct a Hamiltonian formalism for any mechanical system. First let us define the dynamics in the phase space $\mathcal{T}$. Given the 1-form η, one can associate to every differentiable function $\mathscr{H} : \mathcal{T} \to \mathbb{R}$, a vector field $X_{\mathscr{H}}$, called the *contact Hamiltonian vector field generated by* $\mathscr{H}$, defined through the two (intrinsic) relations

$$\pounds_{X_{\mathscr{H}}} \eta = f_{\mathscr{H}}\, \eta \qquad \text{and} \qquad -\mathscr{H} = \eta(X_{\mathscr{H}}), \tag{29}$$

where $f_{\mathscr{H}} \in C^\infty(\mathcal{T})$ is a function depending on $\mathscr{H}$ to be fixed below, cf. Eqs. (33) and (35), and $\mathscr{H}$ is called the *contact Hamiltonian* [19,28,39]. The first condition in (29) means that $X_{\mathscr{H}}$ generates a contact transformation (see Section 3.3 below), while the second condition guarantees that it is generated by a Hamiltonian function. Using Cartan's identity [14]

$$\pounds_{X_{\mathscr{H}}} \eta = \mathrm{d}\eta(X_{\mathscr{H}}) + \mathrm{d}[\eta(X_{\mathscr{H}})] \tag{30}$$

and the second condition in (29), it follows that

$$\mathrm{d}\mathscr{H} = \mathrm{d}\eta(X_{\mathscr{H}}) - \pounds_{X_{\mathscr{H}}} \eta, \tag{31}$$

from which it is clear that the definition of a contact Hamiltonian vector field generalizes that of a symplectic Hamiltonian vector field to the case where the defining 1-form is not preserved along the flow [cf. Eqs. (31) and (1)].

An example of a contact manifold is the extended phase space Γ^{E} that we have introduced in Section 2.3 in order to account for time-dependent Hamiltonian systems. In fact, it is easy to prove that the Poincaré-Cartan 1-form η_{PC} satisfies the condition (28) and therefore it defines a contact structure on Γ^{E}.

Associated with the definition of the contact 1-form on a contact manifold, there is another fundamental object called the *Reeb vector field* ξ, which is defined intrinsically by the conditions

$$\eta(\xi) = 1, \qquad \mathrm{d}\eta(\xi) = 0. \tag{32}$$

It can be shown that such vector field is unique and that it defines at every point a 'vertical' direction with respect to the horizontal distribution $\mathcal{D} = \ker(\eta)$. Finally, using (31) and (32), it is easy to prove that the two conditions in (29) imply

$$f_{\mathscr{H}} = -\xi(\mathscr{H}). \tag{33}$$

It is always possible to find a set of local (Darboux) coordinates (q^a, p_a, S) for $\mathcal{T}$ [14], to which we refer as *contact coordinates*, such that the 1-form η and the Reeb vector field ξ can be written as

$$\eta = \mathrm{d}S - p_a \mathrm{d}q^a, \qquad \xi = \frac{\partial}{\partial S}. \tag{34}$$

We remark that η as in (34) is the standard (natural) contactification of a symplectic manifold whose symplectic structure is exact, as defined e.g. in [28] and that the second expression in (34) directly implies that in these coordinates

$$f_{\mathscr{H}} = -\xi(\mathscr{H}) = -\frac{\partial \mathscr{H}}{\partial S}. \tag{35}$$

2 The reader familiar with the geometric representation of quantum mechanics might notice the similarity between the concepts of contact phase space and quantum phase space. Both of them may be seen as a fiber bundle over the symplectic phase space [35–38].

Besides, in these coordinates, the contact Hamiltonian vector field $X_{\mathscr{H}}$ takes the form

$$X_{\mathscr{H}} = \left(p_a \frac{\partial \mathscr{H}}{\partial p_a} - \mathscr{H}\right)\frac{\partial}{\partial S} - \left(p_a \frac{\partial \mathscr{H}}{\partial S} + \frac{\partial \mathscr{H}}{\partial q^a}\right)\frac{\partial}{\partial p_a} + \left(\frac{\partial \mathscr{H}}{\partial p_a}\right)\frac{\partial}{\partial q^a}. \tag{36}$$

According to Eq. (36), the flow of $X_{\mathscr{H}}$ can be explicitly written in contact coordinates as

$$\dot{q}^a = \frac{\partial \mathscr{H}}{\partial p_a}, \tag{37}$$

$$\dot{p}_a = -\frac{\partial \mathscr{H}}{\partial q^a} - p_a \frac{\partial \mathscr{H}}{\partial S}, \tag{38}$$

$$\dot{S} = p_a \frac{\partial \mathscr{H}}{\partial p_a} - \mathscr{H}. \tag{39}$$

The similarity of Eq. (37)–(39) with Hamilton's equations of symplectic mechanics (5) is manifest. In fact, these are the generalization of Hamilton's equations to a contact manifold. In particular, when $\mathscr{H}$ does not depend on S, Eqs. (37) and (38) give exactly Hamilton's equations in the symplectic phase space and $\mathscr{H}$ is an integral of motion. Finally, the remaining Eq. (39) in this case is the usual definition of Hamilton's principal function—cf. Eq. (26). Therefore (37)–(39) generalize the equations of motion for the positions, the momenta and Hamilton's principal function of the standard Hamilton's theory and can include a much larger class of models, such as the dynamics of basic dissipative systems (that we consider below) and that of systems in equilibrium with a heat bath, i.e. the so-called "thermostatted dynamics" [27,40–42] (see also [22–24] for applications to non-equilibrium thermodynamics).

As an example, given the (1-dimensional) contact Hamiltonian system

$$\mathscr{H}_S = \frac{p^2}{2m} + V(q) + \gamma S \tag{40}$$

where $V(q)$ is the mechanical potential and γ is a constant parameter, the equations of motion (37)–(39) read

$$\dot{q} = \frac{p}{m}, \tag{41}$$

$$\dot{p} = -\frac{\partial V(q)}{\partial q} - \gamma p, \tag{42}$$

$$\dot{S} = \frac{p^2}{2m} - V(q) - \gamma S. \tag{43}$$

From (41) and (42) it is easy to derive the damped Newtonian equation (23), which describes all systems with a friction force that depends linearly on the velocity. Notice that the derivation through the use of contact Hamiltonian dynamics guarantees that the canonical and physical momenta and positions coincide, contrary to what happens in the case of a description by means of explicit time dependence, as for instance in the Caldirola–Kanai model (21).

Before concluding this section, let us remark an important difference between our approach and previous uses of contact geometry to describe non-conservative systems. As we showed in Section 2.3, the evolution of a non-conservative mechanical system whose Hamiltonian depends explicitly on time is usually given in the extended phase space Γ^{E}, endowed with the Poincaré-Cartan 1-form (12), which provides the contact 1-form for Γ^{E} [13]. Usual treatments of time-dependent mechanical systems give the dynamics as in (14). Therefore, according to (29) one finds that the corresponding contact Hamiltonian is

$$-\mathscr{H} = \eta_{\mathrm{PC}}(X_H^{\mathrm{E}}) = p_a \frac{\partial H}{\partial p_a} - H = [H], \tag{44}$$

where $[H]$ stands for the total Legendre transform of H [14]. Moreover, from the condition (13), defining the Hamiltonian dynamics in Γ^{E}, and from the definition of the Reeb vector field (32), one finds immediately that X_H^{E} is proportional to the Reeb vector field ξ in the extended phase space, the proportionality being given by $-\mathscr{H}$. One concludes then that any time-dependent mechanical system can be described in Γ^{E} by the contact Hamiltonian vector field

$$X_{\mathscr{H}} = -\mathscr{H}\xi = \left(p_a \frac{\partial H}{\partial p_a} - H\right) \frac{\partial}{\partial S}. \tag{45}$$

The flow of this vector field in contact coordinates (Q^a, P_a, S) is

$$\dot{Q}^a = 0 \tag{46}$$

$$\dot{P}_a = 0 \tag{47}$$

$$\dot{S} = [H], \tag{48}$$

which coincides with the flow (25)–(26), i.e. the natural evolution in the adapted coordinates found after performing the proper (Hamilton–Jacobi) time-dependent canonical transformation [18].

In this work we decide not to take this description for time-dependent Hamiltonian systems. In fact, we always consider here time-independent symplectic systems as embedded into the contact phase space $\mathcal{T}$ and we use the mechanical Hamiltonian $H_{\mathrm{mec}}(q^a, p_a)$ as a contact Hamiltonian to write the equations of motion in the form (37)–(39). It is easy to see that since H_{mec} does not depend on the additional variable S explicitly, the equations of motion thus derived are Hamilton's equations (5) for the time-independent case. In order to consider the time-dependent case, we develop in Section 3.4 a formalism for time-dependent contact Hamiltonian systems and then again we recover standard mechanical systems given by a mechanical Hamiltonian of the type $H_{\mathrm{mec}}(q^a, p_a, t)$ as a particular case of the more general time-dependent contact Hamiltonian evolution, thus obtaining again the correct Eq. (15) as a particular case. The two main advantages of our perspective are that we can always identify the canonical variables (q^a, p_a) with the physical ones and that – as we show below – we can classify mechanical systems as *dissipative* or *non-dissipative* in terms of the contraction of the phase space volume.

3.2. Time evolution of the contact Hamiltonian and mechanical energy

In this section we derive the evolution of the contact Hamiltonian and the mechanical energy for a system evolving according to the contact Hamiltonian equations (37)–(39) and we show that there is a constant of motion that can help to simplify the solution of the dynamics in particular cases.

Given any function in the contact phase space $\mathscr{F} \in C^\infty(\mathcal{T})$, its evolution according to Eqs. (37)–(39) is given by

$$\begin{aligned} \frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} &= X_{\mathscr{H}}[\mathscr{F}] \\ &= -\mathscr{H}\frac{\partial \mathscr{F}}{\partial S} + p_a\left[\frac{\partial \mathscr{F}}{\partial S}\frac{\partial \mathscr{H}}{\partial p_a} - \frac{\partial \mathscr{F}}{\partial p_a}\frac{\partial \mathscr{H}}{\partial S}\right] + \frac{\partial \mathscr{F}}{\partial q^a}\frac{\partial \mathscr{H}}{\partial p_a} - \frac{\partial \mathscr{F}}{\partial p_a}\frac{\partial \mathscr{H}}{\partial q^a} \\ &= -\mathscr{H}\frac{\partial \mathscr{F}}{\partial S} + p_a\{\mathscr{F}, \mathscr{H}\}_{(S,p_a)} + \{\mathscr{F}, \mathscr{H}\}_{(q^a,p_a)}, \end{aligned} \tag{49}$$

where $\{,\}_{(q^a,p_a)}$ is the standard Poisson bracket as in (7) and the remaining terms are contact corrections. We point out that the bracket $\{,\}_{(S,p_a)}$ is just a shorthand notation and we do not provide any intrinsic definition for it. We say that a function $\mathscr{F} \in C^\infty(\mathcal{T})$ is a *first integral* (or *invariant*) of the contact dynamics given by $X_{\mathscr{H}}$ if $\mathscr{F}$ is constant along the flow of $X_{\mathscr{H}}$, that is if $X_{\mathscr{H}}[\mathscr{F}] = 0$. From the above equations, it follows that the evolution of the contact Hamiltonian function along its flow is

$$\frac{\mathrm{d}\mathscr{H}}{\mathrm{d}t} = -\mathscr{H}\frac{\partial \mathscr{H}}{\partial S}. \tag{50}$$

Therefore in general $\mathscr{H}$ is constant if and only if $\mathscr{H} = 0$ or if $\mathscr{H}$ does not depend on S. The latter case corresponds to a non-dissipative mechanical system, for which $\mathscr{H} = H_{\mathrm{mec}}(q^a, p_a)$ and thus the mechanical energy is conserved. Let us consider a more general case, in which

$$\mathscr{H} = H_{\mathrm{mec}}(q^a, p_a) + h(S), \tag{51}$$

where $H_{\mathrm{mec}}(q^a, p_a)$ is the mechanical energy of the system and $h(S)$ characterizes effectively the interaction with the environment. From (49), the evolution of the mechanical energy is

$$\frac{\mathrm{d}H_{\mathrm{mec}}}{\mathrm{d}t} = -p_a \frac{\partial H_{\mathrm{mec}}}{\partial p_a} h'(S), \tag{52}$$

from which it is clear that h is a potential that generates dissipative forces. For example, in the case of mechanical systems with linear friction represented by the contact Hamiltonian (40), we see that the rate of dissipation of the mechanical energy is

$$\frac{\mathrm{d}H_{\mathrm{mec}}}{\mathrm{d}t} = -m\,\gamma\,\dot{q}^2, \tag{53}$$

which agrees with standard results based on Rayleigh's dissipation function [15]. Furthermore, the evolution of the contact Hamiltonian (51) can be formally obtained from (50) to be

$$\mathscr{H}(t) = \mathscr{H}_0 \exp\left\{-\int_0^t h'(S)\,\mathrm{d}\tau\right\}. \tag{54}$$

In the example of the mechanical system with linear friction, the contact Hamiltonian (40) depends linearly on S and therefore its evolution reads

$$\mathscr{H}_S(t) = \mathscr{H}_{S,0}\, e^{-\gamma t}, \tag{55}$$

where $\mathscr{H}_{S,0}$ is the value of $\mathscr{H}_S$ at $t = 0$.

Eq. (54) introduces the constant of motion $\mathscr{H}_0$, which eliminates one degree of freedom from the equations of motion (37)–(39). In fact, inserting the contact Hamiltonian (51) into (54) one obtains in general

$$H_{\mathrm{mec}}(q^a, p_a) + h(S) = \mathscr{H}_0 \exp\left\{-\int_0^t h'(S)\,\mathrm{d}\tau\right\}, \tag{56}$$

and in principle one can solve this equation for any contact coordinate. In particular, it is possible to solve (56) to obtain S as a function of q^a, p_a and t and therefore the solution of the system (37)–(39) then amounts only to solve the $2n$ equations for the momenta and the positions, as in the standard symplectic case. For example, with $\mathscr{H}_S$ as in (40) one obtains

$$S(q, p, t) = \frac{1}{\gamma}\left[\mathscr{H}_{S,0}\, e^{-\gamma t} - \frac{p^2}{2m} - V(q)\right]. \tag{57}$$

3.3. Contact transformations and Liouville's theorem

In the preceding sections we have introduced the contact phase space for time-independent mechanical systems, equipped with the local coordinates (q^a, p_a, S), called contact coordinates. In these variables the equations of motion are expressed in terms of the contact Hamiltonian equations (37)–(39) and the contact form is expressed as in (34). As in the symplectic case, we are now interested in introducing those transformations that leave the contact structure unchanged, which are known as *contact transformations* [14,39]. Here we consider only time-independent contact transformations and in the next subsection we introduce the time-dependent case.

A *contact transformation* is a transformation that leaves the contact form invariant up to multiplication by a conformal factor [16,17], that is

$$\tilde{\eta} = f\,\eta. \tag{58}$$

From (58), an arbitrary transformation of coordinates from (q^a, p_a, S) to $(\tilde{Q}^a, \tilde{P}_a, \tilde{S})$ is a contact transformation if

$$f(\mathrm{d}S - p_a \mathrm{d}q^a) = \mathrm{d}\tilde{S} - \tilde{P}_a \mathrm{d}\tilde{Q}^a, \tag{59}$$

which is equivalent to

$$f = \frac{\partial \tilde{S}}{\partial S} - \tilde{P}_a \frac{\partial \tilde{Q}^a}{\partial S} \tag{60}$$

$$-fp_i = \frac{\partial \tilde{S}}{\partial q^i} - \tilde{P}_a \frac{\partial \tilde{Q}^a}{\partial q^i} \tag{61}$$

$$0 = \frac{\partial \tilde{S}}{\partial p_i} - \tilde{P}_a \frac{\partial \tilde{Q}^a}{\partial p_i}. \tag{62}$$

As in the standard symplectic theory, we can obtain the generating function of a contact transformation. Assuming that the coordinates $(q^a, \tilde{Q}^a, S)$ are independent, we compute the differential of the generating function $\tilde{S}(q^a, \tilde{Q}^a, S)$, namely

$$\mathrm{d}\tilde{S} = \frac{\partial \tilde{S}}{\partial S}\mathrm{d}S + \frac{\partial \tilde{S}}{\partial q^a}\mathrm{d}q^a + \frac{\partial \tilde{S}}{\partial \tilde{Q}^a}\mathrm{d}\tilde{Q}^a. \tag{63}$$

Substituting (63) into (59) we obtain the following conditions for $\tilde{S}$

$$f = \frac{\partial \tilde{S}}{\partial S}, \quad f\, p_a = -\frac{\partial \tilde{S}}{\partial q^a}, \quad \tilde{P}_a = \frac{\partial \tilde{S}}{\partial \tilde{Q}^a}. \tag{64}$$

In particular, for contact transformations with $f = 1$ the conditions in (64) imply that the generating function has the form

$$\tilde{S} = S - F_1(q^a, \tilde{Q}^a), \tag{65}$$

where $F_1(q^a, \tilde{Q}^a)$ is the generating function of a symplectic canonical transformation, cf. Eq. (10). This result is remarkable, since it implies that all canonical transformations are a special case of contact transformations corresponding to $f = 1$.

While canonical transformations preserve the symplectic volume form Ω^n, we show now that contact transformations induce a re-scaling of the contact volume form $\eta \wedge (\mathrm{d}\eta)^n$. Let us assume that we have a transformation that induces the change $\tilde{\eta} = f\,\eta$; then $\mathrm{d}\tilde{\eta} = \mathrm{d}f \wedge \eta + f\,\mathrm{d}\eta$. It follows that

$$\tilde{\eta} \wedge (\mathrm{d}\tilde{\eta})^n = f^{n+1} \eta \wedge (\mathrm{d}\eta)^n\,, \tag{66}$$

i.e. the volume form is rescaled by a term f^{n+1}, with f given in general in (60). Note that canonical transformations are a special case with $f = 1$ and therefore they preserve the contact volume form.

Finally, applying the contact Hamiltonian vector field $X_{\mathscr{H}}$ to η, we see from (29) and (35) that

$$\pounds_{X_{\mathscr{H}}}\, \eta = f_{\mathscr{H}} \eta = -\frac{\partial \mathscr{H}}{\partial S}\eta. \tag{67}$$

Comparing (67) with (58), we conclude that contact Hamiltonian vector fields are the infinitesimal generators of contact transformations [16,17]. Again, this is the analogue of the fact that symplectic Hamiltonian vector fields are the infinitesimal generators of canonical transformations. Moreover, Eq. (67) also implies that the volume element contracts (or expands) along the contact Hamiltonian flow according to [26]

$$\pounds_{X_{\mathscr{H}}} \left(\eta \wedge (\mathrm{d}\eta)^n\right) = -(n+1)\frac{\partial \mathscr{H}}{\partial S}\left(\eta \wedge (\mathrm{d}\eta)^n\right), \tag{68}$$

which means that the contact flow has a non-zero divergence

$$\mathrm{div}(X_{\mathscr{H}}) = -(n+1)\frac{\partial \mathscr{H}}{\partial S} \tag{69}$$

and therefore Liouville's theorem (11) does not hold. However, an analogous statement of Liouville's theorem for contact flows has been proved in [26]. In fact, although the volume element $\eta\wedge(\mathrm{d}\eta)^n$ is not preserved along the contact Hamiltonian flow, nevertheless a unique invariant measure depending only on $\mathscr{H}$ can be found whenever $\mathscr{H}\neq 0$, given by

$$\mathrm{d}\mu = |\mathscr{H}|^{-(n+1)}\left(\eta\wedge(\mathrm{d}\eta)^n\right), \tag{70}$$

where the absolute value $|\cdot|$ has been introduced in order to ensure that the probability distribution is positive. As it provides an invariant measure for the flow, this is the analogue of Liouville's theorem for contact Hamiltonian flows.

Since the presence of a non-zero divergence is usually interpreted as a sign of dissipation [43,44], here we classify systems as *non-dissipative* or *dissipative* depending on whether the divergence (69) of the associated dynamics vanishes or not.

3.4. Time-dependent contact Hamiltonian systems

In the preceding sections we have seen that contact Hamiltonian mechanics can account for the dynamics of mechanical systems with dissipation and we have proven some results that extend the symplectic formalism to the contact case. However, so far we have considered only time-independent systems. Now we introduce contact Hamiltonian systems that explicitly depend on time. The results of this and the following sections are all new.

To begin, let us extend the contact phase space by adding the time variable to it. Therefore we have an extended manifold $\mathcal{T}^{\mathrm{E}}=\mathcal{T}\times\mathbb{R}$ with natural coordinates derived from contact coordinates as (q^a, p_a, S, t). Then we extend the contact 1-form (34) to the 1-form

$$\eta^{\mathrm{E}} = \mathrm{d}S - p_a\mathrm{d}q^a + \mathscr{H}\mathrm{d}t, \tag{71}$$

where $\mathscr{H}$ is the contact Hamiltonian, that in this case is allowed to depend on t too. Notice that whenever $\mathscr{H}$ depends on S, $\mathrm{d}\eta^{\mathrm{E}}$ is non-degenerate (and closed) and therefore $(\mathcal{T}^{\mathrm{E}}, \mathrm{d}\eta^{\mathrm{E}})$ is a symplectification of $(\mathcal{T}, \eta)$. However, such symplectification is not the standard (natural) one defined e.g. in [28]. Our symplectification depends on the Hamiltonian of the system as it is clear from Eq. (71). This is the same as it happens with the contactification of the symplectic phase space given by the Poincaré-Cartan 1-form (12). Besides, the coordinates (q^a, p_a, S, t) are non-canonical coordinates for $\mathrm{d}\eta^{\mathrm{E}}$, as it is easy to check.

Now we want to define the dynamics on $\mathcal{T}^{\mathrm{E}}$. To do so, we set the two (intrinsic) simultaneous conditions

$$\pounds_{X^{\mathrm{E}}_{\mathscr{H}}}\eta^{\mathrm{E}} = g_{\mathscr{H}}\,\eta^{\mathrm{E}} \qquad\text{and}\qquad \eta^{\mathrm{E}}\left(X^{\mathrm{E}}_{\mathscr{H}}\right)=0, \tag{72}$$

with $g_{\mathscr{H}}\in C^{\infty}(\mathcal{T}^{\mathrm{E}})$ a function depending on $\mathscr{H}$ to be fixed below, cf. Eq. (77). Notice that (72) is the natural extension of (29) to $\mathcal{T}^{\mathrm{E}}$. We argue that these two conditions define a vector field $X^{\mathrm{E}}_{\mathscr{H}}$ on $\mathcal{T}^{\mathrm{E}}$ which is completely equivalent to the contact Hamiltonian flow (36). To prove this, let us first use Cartan's identity (30) to re-write (72) as

$$\mathrm{d}\eta^{\mathrm{E}}(X^{\mathrm{E}}_{\mathscr{H}}) = g_{\mathscr{H}}\eta^{\mathrm{E}} \qquad\text{and}\qquad \eta^{\mathrm{E}}(X^{\mathrm{E}}_{\mathscr{H}})=0. \tag{73}$$

Then we use the second condition in (73). In local coordinates we can write this condition as

$$\left(\mathrm{d}S - p_a\mathrm{d}q^a + \mathscr{H}\mathrm{d}t\right)\left(X^S\frac{\partial}{\partial S} + X^{q^a}\frac{\partial}{\partial q^a} + X^{p_a}\frac{\partial}{\partial p_a} + X^t\frac{\partial}{\partial t}\right) = 0, \tag{74}$$

where the X^i are the general components of the vector field $X^{\mathrm{E}}_{\mathscr{H}}$ in these coordinates. We are free to fix a normalization for $X^{\mathrm{E}}_{\mathscr{H}}$ such that $X^t=1$. Now condition (74) yields

$$X^S = p_a X^{q^a} - \mathscr{H}. \tag{75}$$

Using (75) we can write the first condition in (73) as

$$\mathrm{d}\eta^{\mathrm{E}}\left([p_a X^{q^a} - \mathscr{H}]\frac{\partial}{\partial S} + X^{q^a}\frac{\partial}{\partial q^a} + X^{p_a}\frac{\partial}{\partial p_a} + \frac{\partial}{\partial t}\right) = g_{\mathscr{H}}\eta^{\mathrm{E}}, \tag{76}$$

and, after a direct calculation, one arrives at

$$g_{\mathscr{H}} = -\frac{\partial \mathscr{H}}{\partial S}, \quad X^{q^a} = \frac{\partial \mathscr{H}}{\partial p_a}, \quad X^{p_a} = -\frac{\partial \mathscr{H}}{\partial q^a} - p_a\frac{\partial \mathscr{H}}{\partial S}. \tag{77}$$

Finally, considering all the above conditions, we can write the resulting vector field $X^{\mathrm{E}}_{\mathscr{H}}$ satisfying both conditions in (73) in its general form as

$$X^{\mathrm{E}}_{\mathscr{H}} = X_{\mathscr{H}} + \frac{\partial}{\partial t}, \tag{78}$$

with $X_{\mathscr{H}}$ given by (36). From this it is immediate to recognize that the equations of motion given by such field on $\mathcal{T}^{\mathrm{E}}$ are the same as those of the contact Hamiltonian vector field (36), with the addition of the trivial equation $\dot{t} = 1$. We call a system defined by a contact Hamiltonian $\mathscr{H}(q^a, p_a, S, t)$ and by the vector field $X^{\mathrm{E}}_{\mathscr{H}}$ of the form (78) a *time-dependent contact Hamiltonian system*.[3] From (78) and (49) it follows that the evolution of any function $\mathscr{F} \in C^\infty(\mathcal{T}^{\mathrm{E}})$ under the dynamics given by a time-dependent contact Hamiltonian system reads

$$\frac{\mathrm{d}\mathscr{F}}{\mathrm{d}t} = -\mathscr{H}\frac{\partial \mathscr{F}}{\partial S} + p_a\{\mathscr{F}, \mathscr{H}\}_{(S,p_a)} + \{\mathscr{F}, \mathscr{H}\}_{(q^a,p_a)} + \frac{\partial \mathscr{F}}{\partial t}. \tag{79}$$

Now that we have found a formal prescription to write the equations of motion for time-dependent contact Hamiltonian systems, let us discuss time-dependent contact transformations and their generating functions. *Time-dependent contact transformations* are transformation of coordinates

$$(q^a, p_a, S, t) \to (\tilde{Q}^a, \tilde{P}_a, \tilde{S}, t), \tag{80}$$

that leave the equations of motion, i.e. the vector field $X^{\mathrm{E}}_{\mathscr{H}}$, invariant. By definition, this amounts at finding a transformation that leaves both conditions in (73) unchanged. To find such a transformation, we start with the second condition and write the invariance as the fact that the transformed extended 1-form must have the same form as the original one up to multiplication by a non-zero function f, that is

$$f\left(\mathrm{d}S - p_a\mathrm{d}q^a + \mathscr{H}\mathrm{d}t\right) = \mathrm{d}\tilde{S} - \tilde{P}_a\mathrm{d}\tilde{Q}^a + \mathscr{K}\mathrm{d}t, \tag{81}$$

where $\mathscr{K}$ is a function on $\mathcal{T}^{\mathrm{E}}$ which is going to be the new contact Hamiltonian in the transformed coordinates. This condition provides a way to check whether a transformation of the type (80) is a time-dependent contact transformation. Indeed, inserting the differentials of $\tilde{Q}^a$ and $\tilde{S}$ into (81) one obtains the standard conditions (60)–(62) for a time-independent contact transformation, together with the following rule for the transformation of the Hamiltonians

$$f\,\mathscr{H} = \frac{\partial \tilde{S}}{\partial t} - \tilde{P}_a\frac{\partial \tilde{Q}^a}{\partial t} + \mathscr{K}. \tag{82}$$

As in the time-independent case, in order to find the conditions on the generating function $\tilde{S}(q^a, \tilde{Q}^a, S, t)$ we assume that the coordinates $(q^a, \tilde{Q}^a, S, t)$ are independent. Thus, from (81) one finds that $\tilde{S}$ must satisfy (64) and the additional constraint

$$f\,\mathscr{H} = \frac{\partial \tilde{S}}{\partial t} + \mathscr{K}, \tag{83}$$

[3] We emphasize that, although $(\mathcal{T}^{\mathrm{E}}, \mathrm{d}\eta^{\mathrm{E}})$ is a symplectic manifold, the flow $X^{\mathrm{E}}_{\mathscr{H}}$ has a non-vanishing divergence and therefore it is not a standard symplectic Hamiltonian dynamics, nor it can be introduced in terms of the usual Dirac formalism for time-independent constrained systems.

which defines the new contact Hamiltonian for the new coordinates. In the special case $f = 1$ the generating function reduces to $\tilde{S} = S - F_1(q^a, \tilde{Q}^a, t)$, where $F_1(q^a, \tilde{Q}^a, t)$ is the generating function of the time-dependent canonical transformation, cf. (19).

Now let us consider also the condition on $\tilde{S}$ imposed by invariance of the first equation in (73). Rewriting such equation after the transformation we get

$$\mathrm{d}\tilde{\eta}^{\mathrm{E}}(\tilde{X}^{\mathrm{E}}_{\mathscr{H}}) = \tilde{g}_{\mathscr{K}}\, \tilde{\eta}^{\mathrm{E}} \tag{84}$$

with

$$\tilde{g}_{\mathscr{K}} = -\frac{\partial \mathscr{K}}{\partial \tilde{S}} \tag{85}$$

in the new coordinates, cf. the first condition in (77). Using that $\tilde{\eta}^{\mathrm{E}} = f\eta^{\mathrm{E}}$ and that $\tilde{X}^{\mathrm{E}}_{\mathscr{H}} = X^{\mathrm{E}}_{\mathscr{H}}$ and the two equations in (73), one arrives directly at the following relation

$$f\tilde{g}_{\mathscr{K}} = fg_{\mathscr{H}} - \mathrm{d}f(X^{\mathrm{E}}_{\mathscr{H}}). \tag{86}$$

Notice that for $f = 1$, which corresponds to canonical transformations, (86) reads $\tilde{g}_{\mathscr{K}} = g_{\mathscr{H}}$, from which we infer that if $\mathscr{H}$ does not depend on S, then $\mathscr{K} = 0$ is a possible solution of (86) and in such case (83) reduces to the standard Hamilton–Jacobi transformation (24). However, in the general case f is a function of the extended phase space and thus time-dependent contact transformations extend canonical transformations, as we show with the following example.

To illustrate the formalism developed so far, we consider an example of an important time-dependent contact transformation, i.e. we prove that the Caldirola–Kanai Hamiltonian (21) and the contact Hamiltonian (40) – which both give the same damped Newtonian equation – are related by a time-dependent contact transformation with $f = e^{\gamma t}$. To do so, let us consider the Caldirola–Kanai Hamiltonian H_{CK} as a function on the extended contact phase space $\mathcal{T}^{\mathrm{E}}$ written in the coordinates $(q_{\mathrm{CK}}, p_{\mathrm{CK}}, S_{\mathrm{CK}}, t)$ and the contact Hamiltonian $\mathscr{H}_S$ as a function on $\mathcal{T}^{\mathrm{E}}$ written in the coordinates (q, p, S, t). Defining the change of coordinates [30,32–34]

$$(q, p, S, t) \to (q_{\mathrm{CK}} = q, p_{\mathrm{CK}} = e^{\gamma t}p, S_{\mathrm{CK}} = e^{\gamma t}S, t), \tag{87}$$

it is easy to check that the conditions (60)–(62) and (82) are satisfied and therefore (87) is a time-dependent contact transformation.

3.5. Hamilton–Jacobi formulation

In this section we introduce a Hamilton–Jacobi formulation of contact Hamiltonian systems. This formulation has a major importance, because it establishes a connection with the configuration space, where the phenomenological equations are defined.

The Hamilton–Jacobi equation is a re-formulation of the dynamical equations in terms of a single partial differential equation (PDE) for the function $S(q^a, t)$. Thus, we are looking for a PDE of the form

$$\mathbb{F}\left(q^a, \frac{\partial S}{\partial q^a}, S, t, \frac{\partial S}{\partial t}\right) = 0, \tag{88}$$

whose characteristic curves are equivalent to the contact Hamiltonian dynamics (37)–(39). To construct such PDE, let us define the function

$$\mathbb{F}\left(q^a, p_a, S, t, E\right) \equiv E - \mathscr{H}(q^a, p_a, S, t). \tag{89}$$

It turns out that the solution of the equation $\mathbb{F} = 0$ on the configuration space defined by

$$\eta^{\mathrm{E}} = \mathrm{d}S - p_a \mathrm{d}q^a + \mathscr{H}\mathrm{d}t = 0, \tag{90}$$

that is by the two conditions

$$p_a = \frac{\partial S}{\partial q^a} \quad \text{and} \quad \mathscr{H}\left(q^a, \frac{\partial S}{\partial q^a}, S, t\right) = -\frac{\partial S}{\partial t} \tag{91}$$

gives exactly the contact Hamiltonian equations (37)–(39), together with $\dot{t} = 1$ and $\dot{\mathscr{H}} = -\mathscr{H}\partial\mathscr{H}/\partial S + \partial\mathscr{H}/\partial t$, which is the evolution of the time-dependent contact Hamiltonian (79). Therefore we call the second equation in (91) the *contact Hamilton–Jacobi equation*.

In the symplectic case the Hamilton–Jacobi equation is also the time-dependent canonical transformation induced by the Hamiltonian dynamics. To find an equivalent formulation for the contact case, one must find a generating function $\tilde{S}(q^a, Q^a, S, t)$ such that (83) reduces to (91), where from (67) the function f is

$$f = \exp\left(-\int_0^t \frac{\partial\mathscr{H}}{\partial S}\mathrm{d}\tau\right). \tag{92}$$

However, contrary to the symplectic case, in general such transformation does not lead to a vanishing $\mathscr{K}$, cf. Eq. (86).

In the Appendix B we give an alternative proof of the equivalence between the contact Hamilton–Jacobi equation and (37)–(39). Such proof is useful because it yields explicitly the algebraic conditions needed to recover the solution $q^i(t)$ from knowledge of the complete solution of (91), cf. Eq. (B.10). An example of such procedure is worked out in detail in Section 3.6.3.

3.6. Example: the damped parametric oscillator

In this section we provide an important example, which enables us to show the usefulness of our formalism. The example considered here is the one-dimensional damped parametric oscillator with mass m and time-dependent frequency $\omega(t)$, whose contact Hamiltonian is

$$\mathscr{H} = \frac{p^2}{2m} + \frac{1}{2}m\omega^2(t)q^2 + \gamma S. \tag{93}$$

Clearly the damped harmonic oscillator is obtained for $\omega(t) = \omega_0$ and the damped free particle is recovered when $\omega(t) = 0$. The dynamics of the system is given by the contact Hamiltonian equations (41)–(43), with the time-dependent potential $V = \frac{1}{2}m\omega^2(t)q^2$. Our aim is to use the tools of contact geometry to solve the dynamics. We show three different ways to solve this system, the first of them using contact transformations, the second one using the integrals of motion and the last one by means of the contact Hamilton–Jacobi equation.

3.6.1. First route to the solution: contact transformations

In this section we show how to use time-dependent contact transformations to reduce the system to a known form and thus find a solution. Let us start by introducing the contact transformation

$$(q_{\mathrm{E}}, p_{\mathrm{E}}, S_{\mathrm{E}}, t) = \left(q\, e^{\frac{\gamma t}{2}}, \left[p + \frac{m\gamma}{2}q\right] e^{\frac{\gamma t}{2}}, \left[S + \frac{m\gamma}{4}q^2\right] e^{\gamma t}, t\right). \tag{94}$$

The new coordinates q_{E}, p_{E} and S_{E} are known in the literature as the *expanding coordinates* [32–34]. The new Hamiltonian in these coordinates is obtained from (82) to be

$$\mathscr{H}_{\mathrm{E}} = e^{\gamma t}\mathscr{H} - \frac{\partial S_{\mathrm{E}}}{\partial t} = \frac{p_{\mathrm{E}}^2}{2m} + \frac{m}{2}\left(\omega^2(t) - \frac{\gamma^2}{4}\right) q_{\mathrm{E}}^2. \tag{95}$$

The Hamiltonian $\mathscr{H}_{\mathrm{E}}$ is known as the *expanding Hamiltonian* and represents a parametric oscillator with shifted frequency $\omega^2(t) - \frac{\gamma^2}{4}$. This model has been extensively studied since the sixties and there are many methods to obtain the solutions of the equations of motion, see for example [45–47]. It is interesting to note that in the case $\omega(t) = \omega_0$ the Hamiltonian $\mathscr{H}_{\mathrm{E}}$ itself is an invariant of motion.

3.6.2. Second route to the solution: the invariants

As in the standard symplectic theory, an important tool to solve the contact Hamiltonian equations are the invariants (or first integrals) of the system, which are functions of the (extended) contact phase space that do not vary along the flow, cf. Eq. (79). In Appendix A we prove that the damped parametric oscillator possesses the quadratic invariant

$$\mathscr{I}(q,p,t)=\frac{m\,e^{\gamma\,t}}{2}\left[\left(\alpha(t)\frac{p}{m}-\left[\dot{\alpha}(t)-\frac{\gamma}{2}\alpha(t)\right]q\right)^{2}+\left(\frac{q}{\alpha(t)}\right)^{2}\right],\tag{96}$$

where the purely time-dependent function $\alpha(t)$ satisfies the *Ermakov equation*

$$\ddot{\alpha}+\left(\omega^{2}(t)-\frac{\gamma^{2}}{4}\right)\alpha=\frac{1}{\alpha^{3}},\tag{97}$$

and the S-dependent invariant

$$\mathscr{G}(q,p,S,t)=e^{\gamma t}\left[S-\frac{q\,p}{2}\right].\tag{98}$$

The invariant $\mathscr{I}(q,p,t)$ is a generalization of the canonical invariant found by H. R. Lewis Jr. for the parametric oscillator [45], which is recovered when $\gamma\to 0$. Besides, the invariant $\mathscr{G}$ is completely new.

To solve the equations of motion of the system (93) in the general case, we use the invariants $\mathscr{I}$ and $\mathscr{G}$ to define the time-dependent contact transformation

$$\tilde{Q}=\arctan\left(\alpha\left[\dot{\alpha}-\frac{\gamma}{2}\alpha\right]-\alpha^{2}\frac{p}{m\,q}\right)\tag{99}$$

$$\tilde{P}=\mathscr{I}(q,p,t)\tag{100}$$

$$\tilde{S}=\mathscr{G}(q,p,S,t)\tag{101}$$

$$t=t.\tag{102}$$

The conformal factor in Eq. (81) for this transformation is $f=e^{\gamma t}$ and the new contact Hamiltonian, from Eq. (82), takes the simple form

$$\mathscr{K}=\frac{\mathscr{I}}{\alpha^{2}}.\tag{103}$$

Thus, as $\mathscr{K}$ does not involve the variables $\tilde{Q}$ and $\tilde{S}$, the new contact Hamiltonian equations have the trivial form

$$\dot{\tilde{Q}}^{a}=\frac{1}{\alpha^{2}},\quad \dot{\tilde{P}}_{a}=0,\quad \dot{\tilde{S}}=0,\tag{104}$$

with solutions

$$\tilde{Q}(t)=\int^{t}\frac{\mathrm{d}\tau}{\alpha^{2}(\tau)},\quad \tilde{P}(t)=\mathscr{I}\quad\text{and}\quad \tilde{S}(t)=\mathscr{G}.\tag{105}$$

Now, inverting the transformation (99)–(101) and using (105), one obtains the solutions in the original (physical) coordinates, namely

$$q(t)=\sqrt{\frac{2\mathscr{I}}{m}}e^{\gamma t}\alpha(t)\cos\phi(t),\tag{106}$$

$$p(t)=\sqrt{2\,m\mathscr{I}}\,e^{\gamma t}\left[\left(\dot{\alpha}-\frac{\gamma}{2}\alpha\right)\cos\phi(t)-\frac{1}{\alpha}\sin\phi(t)\right],\tag{107}$$

$$S(t)=\mathrm{e}^{-\gamma t}\mathscr{G}+\frac{q(t)p(t)}{2},\tag{108}$$

where $\phi(t)=\tilde{Q}(t)$ and the values of the constants $\mathscr{I}$ and $\mathscr{G}$ are determined by the initial conditions. Therefore, we have derived here the solutions of the equations of motion of the damped parametric oscillator using the invariants of the contact Hamiltonian system and a proper contact transformation. From (106)–(108) we see that all the dynamics of the system is encoded in the Ermakov equation (97).

3.6.3. Third route to the solution: the contact Hamilton–Jacobi equation

We show here another way to find the evolution of the system (93), that is, by solving the corresponding contact Hamilton–Jacobi equation (91), which in this case reads

$$\frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2+\frac{1}{2}m\omega^2(t)q^2+\gamma S=-\frac{\partial S}{\partial t}\,. \tag{109}$$

Due to the form of the left hand side of the above equation, one can propose that $S(q,t)$ is a polynomial with respect to q. Thus we choose the ansatz

$$S(q,t)=m\,C(t)\left[\frac{q^2}{2}-\lambda(t)q\right]+mq\dot{\lambda}+s(t), \tag{110}$$

where $C(t)$, $\lambda(t)$ and $s(t)$ are purely time-dependent functions. It follows directly that

$$p(q,t)=\frac{\partial S}{\partial q}=m\,C(t)\,[q-\lambda(t)]+m\dot{\lambda}(t). \tag{111}$$

Besides, inserting $S(q,t)$ into the contact Hamilton–Jacobi equation (109) and comparing the coefficients of the same order in q, we can find the conditions on $C(t)$, $\lambda(t)$ and $s(t)$. After a direct calculation one obtains that $C(t)$ obeys the *Riccati equation*

$$\dot{C}+C^2+\gamma\,C+\omega^2(t)=0, \tag{112}$$

$\lambda(t)$ satisfies the damped Newtonian equation

$$\ddot{\lambda}+\gamma\,\dot{\lambda}+\omega^2(t)\,\lambda=0 \tag{113}$$

and

$$\dot{s}=-\frac{m}{2}\left[C^2\lambda^2-2C\lambda\dot{\lambda}+\dot{\lambda}^2\right]-\gamma s. \tag{114}$$

Now one can use the Riccati equation (112) and the Newton equation (113) to integrate (114) and obtain

$$s(t)=\frac{m}{2}\left[C(t)\lambda^2(t)-\lambda(t)\dot{\lambda}(t)\right]. \tag{115}$$

Substituting into (110), one finds that the solution of the contact Hamilton–Jacobi equation (109) is

$$S(q,t)=\frac{m}{2}C(t)\,[q-\lambda(t)]^2+m\dot{\lambda}(t)\,[q-\lambda(t)]+\frac{m}{2}\lambda(t)\dot{\lambda}(t). \tag{116}$$

Let us mention that solutions $C(t)$ of the Riccati equation are connected to solutions $\lambda(t)$ of the damped Newton equation by means of the transformation $C(t)=\dot{\lambda}(t)/\lambda(t)$. Therefore, in order to determine $S(q,t)$ it is sufficient to solve only one of these equations.

Now given the solution (116) of the contact Hamilton–Jacobi equation (91), depending on the non-additive constant $C_0=C(0)$, the trajectory of the particle $q(t)$ can be obtained using (B.3) and (B.10) as follows. For this system (B.10) implies $b(t)=b_0\,e^{-\gamma t}$ and therefore (B.3) reads

$$\frac{\partial S}{\partial C_0}=\frac{m}{2}\frac{\partial C}{\partial C_0}q^2=b_0\,e^{-\gamma t}, \tag{117}$$

which can be inverted to find the solution trajectory

$$q(t)=\sqrt{\frac{2\,b_0\,e^{-\gamma t}}{m}\left(\frac{\partial C}{\partial C_0}\right)^{-1}}\,. \tag{118}$$

For instance, in the case of a damped free particle, $\omega(t)=0$, the solution of the Riccati equation is

$$C(t)=\frac{e^{-\gamma t}}{\frac{1}{C_0}+\frac{1}{\gamma}\left(1-e^{-\gamma t}\right)}, \tag{119}$$

and from (118) we can recover the correct trajectories

$$q(t) = \sqrt{\frac{2b_0}{m}} \left[1 + \frac{C_0}{\gamma} \left(1 - e^{-\gamma t} \right) \right], \tag{120}$$

where the constants b_0 and C_0 are related to the initial conditions via

$$b_0 = \frac{m}{2} q_0^2 \qquad \text{and} \qquad C_0 = \dot{q}_0. \tag{121}$$

Interestingly, with the method presented in this section the evolution of the particle is ultimately determined by the solution of the Riccati equation (112), while with the method given in Section 3.6.1 one has to directly solve the Newton equation arising from $\mathscr{H}_{\mathrm{E}}$ and with the method introduced in Section 3.6.2 the solution is given in terms of the solution of the Ermakov equation (97). This shows that the three methods presented here within the framework of contact geometry are related to the three standard techniques for the solution of this system.

4. Conclusions and perspectives

In this work we have proposed a new geometric perspective for the Hamiltonian description of mechanical systems. The defining features of our formulation are that the phase space of any (dissipative or non-dissipative) mechanical system is assumed to be a contact manifold and that the evolution equations are given as contact Hamiltonian equations, see (37)–(39). We have shown that contact Hamiltonian dynamics on the one hand recovers all the results of standard symplectic dynamics when the contact Hamiltonian $\mathscr{H}$ does not depend explicitly on S and on the other hand can account for the evolution of systems with different types of dissipation in the more general case in which $\mathscr{H}$ depends on S.

We have considered both time-independent and time-dependent contact systems and we have found in both cases the transformations (called contact transformations) that leave the contact Hamiltonian equations invariant, showing that canonical transformations of symplectic dynamics are a special case. To show the usefulness of contact transformations, we have provided an explicit example (the Caldirola–Kanai model for systems with linear dissipation) in which a non-canonical but contact transformation (87) allows to move from the usual time-dependent canonical description in terms of non-physical variables to a contact description in terms of the physical variables.

By computing the divergence of the contact Hamiltonian flow (69), we have provided a formal definition of dissipation in our formalism in terms of the contraction of the phase space, which is usually associated with irreversible entropy production [43,44].

In addition, we have derived a contact Hamilton–Jacobi equation (91) whose complete solution is equivalent to solve the Hamiltonian dynamics, as it happens in standard symplectic mechanics.

Finally, we have worked out in detail a specific important example (the damped parametric oscillator) for which we have solved the dynamics in three different ways: using contact transformations, using the invariants of the system and resorting to the solution of the associated contact Hamilton–Jacobi equation. This example thus provides a direct evidence of the usefulness of our formalism.

Given the importance of the symplectic perspective in the classical mechanics of conservative systems, we consider that the contact perspective could play a similar role in the mechanics of dissipative systems. For instance, a relevant question is that of a quantization of our formalism. Here we sketch briefly such possibility. Using the fact that the additional contact variable is a generalization of Hamilton's principal function which satisfies the contact version of the Hamilton–Jacobi equation, and that the canonical momenta and positions in our formalism coincide with the physical ones, we suggest a canonical quantization of the contact Hamiltonian based on the standard rules of canonical quantization, namely

$$p_a \to \frac{\hbar}{i} \frac{\partial}{\partial q^a}, \quad q^a \to \hat{q}^a, \quad S(q^a, t) \to \frac{\hbar}{i} \ln \Psi(q^a, t). \tag{122}$$

Using such rules to quantize the contact Hamiltonian $\mathscr{H}$ and obtain the operator $\hat{\mathscr{H}}$, one can define the "contact Schrödinger equation"

$$i\hbar\frac{\partial\Psi}{\partial t}=\hat{\mathscr{H}}\Psi. \tag{123}$$

This equation has a fundamental property: in the case in which the contact Hamiltonian reduces to a symplectic Hamiltonian (i.e. when $\mathscr{H}$ does not depend on S explicitly) and the dynamics reduces to a standard conservative dynamics, Eq. (123) obviously reduces to the standard linear Schrödinger equation and all the known results for the quantization of conservative systems are recovered. However, this equation has the disadvantage that in general it does not conserve the norm of the wave function [32]. For systems with contact Hamiltonian of the form $\mathscr{H}=H_{\rm mec}+h(S)$, see (51), normalization is achieved following the procedure of Gisin [48], which consists in subtracting the mean value of h, that is $\langle\hat{h}\rangle=\int\Psi^*\,\hat{h}\,\Psi\,{\rm d}q^a$. This leads to the nonlinear Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t}=\hat{H}_{\rm mec}\Psi+(\hat{h}-\langle\hat{h}\rangle)\Psi. \tag{124}$$

Applying (124) to the contact Hamiltonian $\mathscr{H}_S$ with linear dependence on S given in Eq. (40) and using (123), we get the evolution equation

$$i\hbar\frac{\partial\Psi}{\partial t}=\left(-\frac{\hbar^2}{2m}\nabla^2+V(q^a)+\gamma\frac{\hbar}{i}\left[\ln\Psi-\langle\ln\Psi\rangle\right]\right)\Psi, \tag{125}$$

which is exactly the phenomenological nonlinear Schrödinger equation introduced in [49–51] for the description of dissipative systems, see also [30,32–34].

This fact, together with the result that the contact dynamics generated by $\mathscr{H}_S$ coincides with the classical Newtonian equations for systems with linear dissipation (23), provides a further theoretical justification for the introduction of the nonlinear phenomenological Schrödinger equation (125) and, most importantly, displays an intriguing consistency between the classical and quantum descriptions in our proposal. A more detailed study of the extension of contact Hamiltonian mechanics to quantum systems will be presented in a future work. For instance, it will be worth trying a more geometric quantization program, e.g. following the lines of [52].

Finally, we have not considered here the Lagrangian formulation. This aspect is fundamental in order to have a complete picture of contact mechanics and it is of primary interest for extension to field theory.

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Appendix A. Invariants for the damped parametric oscillator

In this appendix we prove that $\mathscr{I}(q,p,t)$ and $\mathscr{G}(q,p,S,t)$ given in Eqs. (96) and (98) are two invariants of the damped parametric oscillator defined by the contact Hamiltonian (93). An invariant is a function $\mathscr{F}$ of the (extended) contact phase space that satisfies the partial differential equation

$$-\mathscr{H}\frac{\partial\mathscr{F}}{\partial S}+p_a\{\mathscr{F},\mathscr{H}\}_{(S,p_a)}+\{\mathscr{F},\mathscr{H}\}_{(q^a,p_a)}=-\frac{\partial\mathscr{F}}{\partial t}, \tag{A.1}$$

where we use the same notation as in (49). To find a solution, we propose the ansatz

$$\mathscr{F}(q,p,S,t)=\beta(t)p^2-2\xi(t)qp+\eta(t)q^2+\zeta(t)S. \tag{A.2}$$

Inserting (A.2) into (A.1), we get the system of ordinary differential equations

$$\dot\beta = \frac{2}{m}\xi + 2\gamma\beta - \frac{1}{2m}\zeta, \tag{A.3}$$

$$\dot\eta = -2m\omega^2\xi + \frac{1}{2}m\omega^2\zeta, \tag{A.4}$$

$$\dot\xi = \frac{1}{m}\eta + \gamma\xi - m\omega^2\beta, \tag{A.5}$$

$$\dot\zeta = \gamma\,\zeta. \tag{A.6}$$

Then clearly

$$\zeta(t) = \zeta_0 e^{\gamma t}, \tag{A.7}$$

and we are left with the problem of solving the system (A.3)–(A.5). To do so, we consider the change of variables $\tilde\beta(t) = e^{-\gamma t}\beta(t)$, $\tilde\eta(t) = e^{-\gamma t}\eta(t)$ and $\tilde\xi(t) = e^{-\gamma t}\xi(t)$, which yields the equivalent system

$$\dot{\tilde\beta} = \frac{2}{m}\tilde\xi + \gamma\tilde\beta - \frac{\zeta_0}{2m}, \tag{A.8}$$

$$\dot{\tilde\eta} = -2m\omega\tilde\xi - \gamma\tilde\eta + \frac{\zeta_0}{2}m\omega^2, \tag{A.9}$$

$$\dot{\tilde\xi} = \frac{1}{m}\tilde\eta - m\omega^2\tilde\beta. \tag{A.10}$$

To solve this system, we re-write it as a third-order ordinary differential equation for $\tilde\beta(t)$

$$\dddot{\tilde\beta} + 4\,\Omega^2\,\dot{\tilde\beta} + 4\Omega\,\dot\Omega\,\tilde\beta = 0, \tag{A.11}$$

where for simplicity we define $\Omega^2 = \omega^2 - \frac{\gamma^2}{4}$. The above equation is known as *the normal form* of a third order equation of maximal symmetry [53].

Now, using the further change of variable

$$\tilde\beta(t) = \frac{1}{2m}\alpha^2(t) \tag{A.12}$$

in (A.11), one obtains that $\tilde\beta(t)$ is a solution of (A.11) if and only if $\alpha(t)$ is a solution of the Ermakov equation (97). Moreover, from (A.12) one can re-write the remaining two equations as

$$\tilde\eta(t) = \frac{m}{2}\left(\left[\dot\alpha(t) - \frac{\gamma}{2}\alpha(t)\right]^2 + \frac{1}{\alpha^2(t)}\right), \tag{A.13}$$

$$\tilde\xi(t) = \frac{\alpha(t)}{2}\left(\dot\alpha(t) - \frac{\gamma}{2}\alpha(t)\right) + \frac{1}{4}. \tag{A.14}$$

Finally, using (A.7), (A.12)–(A.14) and $\beta(t) = e^{\gamma t}\tilde\beta(t)$, $\eta(t) = e^{\gamma t}\tilde\eta(t)$, $\xi(t) = e^{\gamma t}\tilde\xi(t)$ into the ansatz (A.2), we find that

$$\mathscr{F}(q,p,S,t) = \mathscr{I}(q,p,t) + \zeta_0\mathscr{G}(q,p,S,t), \tag{A.15}$$

with

$$\mathscr{I}(q,p,t) = \frac{m\,e^{\gamma t}}{2}\left[\left(\alpha(t)\frac{p}{m} - \left[\dot\alpha(t) - \frac{\gamma}{2}\alpha(t)\right]q\right)^2 + \left(\frac{q}{\alpha(t)}\right)^2\right], \tag{A.16}$$

and

$$\mathscr{G}(q,p,S,t) = e^{\gamma t}\left[S - \frac{q(t)p(t)}{2}\right]. \tag{A.17}$$

Since $\mathscr{F}(q,p,S,t)$ is an invariant for any choice of the initial conditions and since ζ_0 only depends on the initial conditions, it follows that $\mathscr{I}(q,p,t)$ and $\mathscr{G}(q,p,S,t)$ separately are invariants of the system.

Appendix B. Equivalence between the contact Hamilton–Jacobi equation and the contact Hamiltonian equations

In this appendix we prove that finding the complete solution of the contact Hamilton–Jacobi equation (91) is equivalent to solving the equations of motion (37)–(39). This proof is a generalization of the standard proof for the symplectic case [15].

To begin, let $S(q^1, \ldots, q^n, c^1, \ldots, c^n, t)$ be the complete solution of (91), where c^i are n constants and suppose

$$\left|\frac{\partial^2 S}{\partial q^i \partial c^j}\right| \neq 0. \tag{B.1}$$

Using the quantities $p_i(q^1, \ldots, q^n, c^1, \ldots, c^n, t) = \frac{\partial S}{\partial q^i}$, we can rewrite (91) as

$$\mathscr{H}\left(q^1, \ldots, q^n, p_1, \ldots, p_n, S, t\right) = -\frac{\partial S}{\partial t}. \tag{B.2}$$

Besides, defining

$$b_i = \frac{\partial S}{\partial c^i}, \tag{B.3}$$

we obtain

$$\dot{b}_i = \frac{\partial^2 S}{\partial q^j \partial c^i}\dot{q}^j + \frac{\partial^2 S}{\partial t\, \partial c^i} \tag{B.4}$$

and deriving (B.2) with respect to c^i we have

$$\frac{\partial^2 S}{\partial c^i \partial t} = -\frac{\partial \mathscr{H}}{\partial S} b_i - \frac{\partial \mathscr{H}}{\partial p_j}\frac{\partial^2 S}{\partial c^i \partial q^j}. \tag{B.5}$$

Combining (B.4) and (B.5) we get

$$\dot{b}^i = \frac{\partial^2 S}{\partial q^j \partial c^i}\left(\dot{q}^j - \frac{\partial \mathscr{H}}{\partial p_j}\right) - \frac{\partial \mathscr{H}}{\partial S} b_i. \tag{B.6}$$

Now, from the definition of p_i, it follows that

$$\dot{p}_i = \frac{\partial^2 S}{\partial q^j \partial q^i}\dot{q}^j + \frac{\partial^2 S}{\partial t\, \partial q^i}, \tag{B.7}$$

and deriving (B.2) with respect to q^i one obtains

$$\frac{\partial^2 S}{\partial q^i \partial t} = -\frac{\partial \mathscr{H}}{\partial q^i} - \frac{\partial \mathscr{H}}{\partial S} p_i - \frac{\partial \mathscr{H}}{\partial p_j}\frac{\partial^2 S}{\partial q^i \partial q^j}. \tag{B.8}$$

From (B.7) and (B.8) we get

$$\dot{p}_i = \frac{\partial^2 S}{\partial q^j \partial q^i}\left(\dot{q}^j - \frac{\partial \mathscr{H}}{\partial p_j}\right) - \frac{\partial \mathscr{H}}{\partial q^i} - p_i \frac{\partial \mathscr{H}}{\partial S}. \tag{B.9}$$

It is thus easy to see that imposing

$$\dot{b}_i = -\frac{\partial \mathscr{H}}{\partial S} b_i, \tag{B.10}$$

Eqs. (B.6) and (B.9) reduce to

$$\dot{q}^i = \frac{\partial \mathscr{H}}{\partial p_i}, \tag{B.11}$$

$$\dot{p}_i = -\frac{\partial \mathscr{H}}{\partial q^i} - p_i \frac{\partial \mathscr{H}}{\partial S}, \tag{B.12}$$

which coincide with (37) and (38). Finally, using the fact that c^i are constants of motion, the equation for the evolution of S reads

$$\dot{S} = \frac{\partial S}{\partial q^i}\dot{q}^i + \frac{\partial S}{\partial t} = p_i \frac{\partial \mathscr{H}}{\partial p_i} - \mathscr{H}, \tag{B.13}$$

where in the last equality we have used that $p_i = \partial S/\partial q^i$, $\dot{q}^i = \partial \mathscr{H}/\partial p_i$ and that $\mathscr{H} = -\partial S/\partial t$. Eqs. (B.11)–(B.13) are exactly equivalent to (37)–(39). Therefore we have proved that the contact Hamilton–Jacobi equation (91) is equivalent to the contact Hamiltonian dynamics, provided the condition (B.10) holds. Therefore such condition has a primary importance. In fact, this yields the algebraic conditions to be solved for q^i in order to recover the solution $q^i(t)$ from knowledge of $S(q^i, c^i, t)$. For an explicit example see Section 3.6.3.

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